Space–time boundary element methods

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based on joint work with

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Heat equation for $(x, t) \in Q := (0, 1) \times (0, T)$

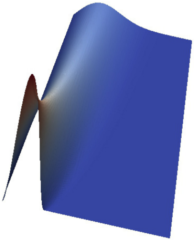
$$\alpha\, u_t(x,t) - u_{xx}(x,t) = 0, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x)$$

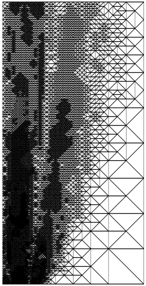
Solution

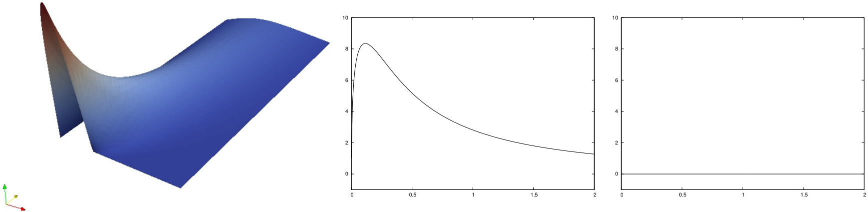
$$u(x,t) = \sum_{k=1}^{\infty} u_k e^{-(k\pi)^2 t/\alpha} \sin k\pi x, \quad u_k = 2\int_0^1 u_0(x) \sin k\pi x\, dx$$

Uniform/adaptive piecewise linear finite element interpolation in Q

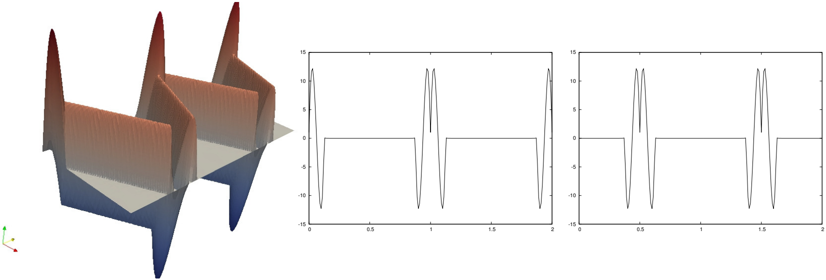
level	nodes	$\|u - I_h u\|_{L^2(Q)}$	eoc
3	153	2.929 –2	
4	561	1.081 –2	1.44
5	2145	2.750 –3	1.97
6	8385	6.870 –4	2.00
7	33153	1.718 –4	2.00
15	9645	1.973 –4	







Solution of the heat equation and normal derivatives at boundary nodes.



Solution of the wave equation and normal derivatives at boundary nodes.

Dirichlet boundary value problem

$$\begin{aligned} \alpha\partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) \quad \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= g(x,t) \quad \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{for } x \in \Omega. \end{aligned}$$

Compatibility condition: $g(x,0) = u_0(x)$ for $x \in \Gamma$.

Variational formulation

Find $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T,H^{-1}(\Omega))$, $u(x,t) = g(x,t)$ for $x \in \Gamma$, $t \in (0,T)$, $u(x,0) = u_0(x)$ for $x \in \Omega$, such that

$$\int_0^T \int_\Omega \Big[\alpha\partial_t u(x,t)v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t)\Big]\, dx\, dt = \int_0^T \int_\Omega f(x,t)v(x,t)\, dx\, dt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Bilinear form

$$a_\Omega(u,v) := \int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t) \Big] \, dx \, dt$$

Consider $u_p \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ satisfying

$$u_p(x,t) = g(x,t) \quad \text{for } x \in \Gamma, \quad t \in (0,T), \quad u_p(x,0) = u_0(x) \quad \text{for } x \in \Omega$$

Primal Formulation: Find $\overline{u} \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ such that

$$a_\Omega(\overline{u} + u_p, v) = \langle f, v \rangle_Q$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Galerkin–Petrov variational formulation;
unique solvability based on inf–sup stability condition
[Schwab, Stevenson 2009; Urban, Patera 2014; Andreev 2013; Mollet 2014; OS 2015]

For $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ find $w \in L^2(0,T;H^1_0(\Omega))$ such that

$$\begin{array}{rcll} -\Delta_x w(x,t) & = & \alpha \partial_t u(x,t) - \Delta_x u(x,t) & \text{in } Q = \Omega \times (0,T), \\ w(x,t) & = & 0 & \text{on } \Sigma = \Gamma \times (0,T). \end{array}$$

Find $w \in L^2(0,T;H^1_0(\Omega))$ such that

$$\int_0^T \int_\Omega \nabla_x w(x,t) \cdot \nabla_x v(x,t)\, dx\, dt = a_\Omega(u,v)$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$

Lemma

$$\|\alpha \partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|w\|^2_{L^2(0,T;H^1_0(\Omega))} = a_\Omega(u,w)$$

Theorem: For $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ we have

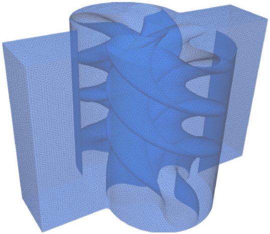
$$\|\alpha\partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} \leq \sup_{0\neq v\in L^2(0,T;H^1_0(\Omega))} \frac{a_\Omega(u,v)}{\|v\|_{L^2(0,T;H^1_0(\Omega))}}$$

Lemma: Norm equivalence for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$

$$\|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} \simeq \|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}$$

Finite element discretization schemes

- space–time tensor product wavelets [Schwab, Stevenson 2009; Urban, Patera 2014; Andreev 2013; Mollet 2014]
- Space–time DG FEM [Neumüller 2013]
- Space–time FEM [OS 2015]



Dual variational formulation

Find $u \in L^2(0,T;H^1(\Omega))$, $u(x,t) = g(x,t)$ for $x \in \Gamma$, $t \in (0,T)$, such that

$$\int_0^T \int_\Omega \Big[-u(x,t)\alpha\partial_t v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t) \Big]\, dx\, dt = \int_0^T \int_\Omega f(x,t) v(x,t)\, dx\, dt + \int_\Omega \alpha u_0(x) v(x,0)\, dx$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))$.

Theorem: For $u \in L^2(0,T;H^1_0(\Omega))$ we have

$$\frac{1}{\sqrt{2}} \|u\|_{L^2(0,T;H^1_0(\Omega))} \le \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))} \frac{\widetilde{a}_\Omega(u,v)}{\|v\|^2_{L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))}}$$

Dirichlet boundary value problem

$$\begin{aligned}
\alpha\partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T),\\
u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T),\\
u(x,0) &= 0 && \text{for } x \in \Omega.
\end{aligned}$$

Primal variational formulation

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)) \to [L^2(0,T;H^1_0(\Omega))]'$$

Dual variational formulation

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \to [L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]'$$

Interpolation, $\theta \in [0,1]$ [Lions, Magenes 1972]

$$\begin{aligned}
\mathcal{L} \quad : \quad & [L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)), L^2(0,T;H^1_0(\Omega))]_\theta\\
& \to [[L^2(0,T;H^1_0(\Omega))]', [L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]']_\theta\\
& = [L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)), L^2(0,T;H^1_0(\Omega))]'_{1-\theta}
\end{aligned}$$

$\theta = \frac{1}{2}$ $\qquad \mathcal{L} : H_0^{1,\frac{1}{2}}(Q) \to [H_0^{1,\frac{1}{2}}(Q)]'$

Dirichlet trace operator [Lions, Magenes 1972]

$$\gamma_0 : H^{1,\frac{1}{2}}(Q) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

Analysis of boundary integral operators

- D. N. Arnold, P. J. Noon 1987, 1989; P. J. Noon 1988
- G. C. Hsiao, J. Saranen 1989, 1993
- M. Costabel 1990
- ...

\+ ellipticity of single layer boundary integral operator

$$V : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

\+ standard analysis of discretization schemes

? results are based on interpolation between primal and dual formulation

? link to Green's formula not obvious

? coupling of finite and boundary element methods

Dirichlet boundary value problem

$$\begin{aligned} \alpha\partial_t u(x,t) - \Delta_x u(x,t) &= 0 && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= g(x,t) && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega. \end{aligned}$$

Representation formula for $x \in \Omega$ and $t \in (0,T)$

$$\begin{aligned} u(x,t) &= \frac{1}{\alpha}\int_0^T \int_\Gamma U^*(x-y,t-s)\partial_{n_y} u(y,s)\, ds_y\, ds \\ &\quad -\frac{1}{\alpha}\int_0^T \int_\Gamma \partial_{n_y} U^*(x-y,t-s) g(y,s)\, ds_y\, ds + \int_\Omega U^*(x-y,t) u_0(y)\, dy \end{aligned}$$

Fundamental solution

$$U^*(x-y,t-s) = \begin{cases} \left(\dfrac{\alpha}{4\pi(t-s)}\right)^{n/2} \exp\left(-\dfrac{\alpha|x-y|^2}{4(t-s)}\right) & \text{for } s \in [0,t), \\ 0 & \text{for } s \geq t. \end{cases}$$

Single layer potential

$$(\widetilde{V}w)(x,t) = \frac{1}{\alpha}\int_0^T \int_\Gamma U^*(x-y,t-s)\partial_{n_y} u(y,s)\, ds_y ds$$

Duality

$$\langle \widetilde{V}w, \psi\rangle_Q = \langle w, \varphi\rangle_\Sigma$$

Adjoint problem

$$\begin{aligned} -\alpha\partial_s\varphi(y,s) - \Delta_y\varphi(y,s) &= \psi(y,s) && \text{for } y \in \Omega, s \in (0,T), \\ \varphi(y,T) &= 0 && \text{for } y \in \Omega \end{aligned}$$

Primal formulation: $\varphi \in L^2(0,T;H^1(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))$

$$\int_0^T \int_\Omega \Big[-\alpha\partial_s\varphi(y,s)\phi(y,s) + \nabla_y\varphi(y,s)\cdot\nabla_y\phi(y,s)\Big] dy\, ds$$

$$= \int_0^T \int_\Omega \psi(y,s)\phi(y,s)\, dy\, ds \quad \text{for all } \phi \in L^2(0,T;H^1_0(\Omega))$$

Single layer potential

$$\widetilde{V} : [\gamma_0 L^2(0,T;H^1(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]' \to L^2(0,T;H^1(\Omega))$$

Dual formulation: $\varphi \in L^2(0,T;H^1(\Omega))$

$$\int_0^T \int_\Omega \Big[\varphi \alpha \partial_s \phi(y,s) + \nabla_y \varphi(y,s) \cdot \nabla_y \phi(y,s) \Big] dy\, ds$$
$$= \int_0^T \int_\Omega \psi(y,s) \phi(y,s)\, dy\, ds \quad \phi \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$$

Single layer potential

$$\widetilde{V} : L^2(0,T;H^{-1/2}(\Gamma)) \to L^2(0,T;H^1(\Omega)) \cap H^1_{0,}(0,T;\widetilde{H}^{-1}(\Omega))$$

Corollary

$$V : L^2(0,T;H^{-1/2}(\Gamma)) \to V_\Sigma := \gamma_0[L^2(0,T;H^1(\Omega)) \cap H^1_{0,}(0,T;\widetilde{H}^{-1}(\Omega))]$$

Single layer potential

$$\widetilde{V} : L^2(0,T;H^{-1/2}(\Gamma)) \to L^2(0,T;H^1(\Omega)) \cap H^1_{0,}(0,T;\widetilde{H}^{-1}(\Omega))$$

Green's formula for $u = \widetilde{V}w$:

$$\int_0^T \int_\Gamma \partial_n u(x,t)\, v(x,t)\, ds_x dt = a_\Omega(u,v)$$

Lemma

$$\|\partial_n u\|_{L^2(0,T;H^{-1/2}(\Gamma))} \leq c_2^a \, \|u\|_{L^2(0,T;H^1(\Omega))\cap H^1(0,T;\widetilde{H}^{-1}(\Omega))}$$

Proof of stability: [Nedelec, Planchard 1973; Hsiao, Wendland 1977; Costabel 1990]

$$\int_0^T \int_\Omega \nabla_x u \cdot \nabla_x v \, dx \, dt + \alpha \int_0^T \int_\Omega \partial_t u \, v \, dx \, dt$$
$$= \int_0^T \int_\Gamma \partial_n u \, v \, ds_x dt + \int_0^T \int_\Omega [\alpha \partial_t u - \Delta_x u] \, v \, dx \, dt$$

For $u = \widetilde{V} w, \quad u_{|\Gamma} = Vw, \quad \partial_n u = (\frac{1}{2} I + K')w$ we obtain

$$a_\Omega(u, v) = \langle (\frac{1}{2} I + K')w, v \rangle_\Sigma$$

Correspondingly,

$$a_{\Omega^c}(u, v) = -\langle (-\frac{1}{2} I + K')w, v \rangle_\Sigma$$

Hence,

$$a_\Omega(u, v) + a_{\Omega^c}(u, v) = \langle w, v \rangle_\Sigma$$

In particular for $v = u$ this finally gives

$$\langle w, Vw \rangle_\Sigma = \int_0^T \int_{\mathbb{R}^n} |\nabla_x u|^2 \, dxdt + \frac{\alpha}{2} \int_{\mathbb{R}^n} [u(T)]^2 \, dx$$

Alternatively, for $u = \widetilde{V}w, \quad u_{|\Gamma} = Vw, \quad \partial_n u = (\frac{1}{2}I + K')w$ we may define

$$-\Delta_x z = c_H \partial_t u \quad \text{in } \mathbb{R}^n \backslash \Gamma, \quad z_{|\Gamma} = 0$$

and consider

$$v = u + z = u + N\partial_t u, \quad v_{|\Gamma} = u_{|\Gamma} = Vw$$

Hence,

$$\begin{aligned} \langle w, Vw \rangle_\Sigma &= a_\Omega(u, u + N\partial_t u) + a_{\Omega^c}(u, u + N\partial_t u) \\ &\geq \frac{1}{2}\Big(\|u\|^2_{L_2(0,T;H^1(\Omega))} + \|\partial_t u\|^2_{L_2(0,T;\widetilde{H}^{-1}(\Omega))} \\ &\qquad + \|u\|^2_{L_2(0,T;H^1(\Omega^c))} + \|\partial_t u\|^2_{L_2(0,T;\widetilde{H}^{-1}(\Omega^c))} \Big) \\ &\geq c_1^V \, \|w\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \end{aligned}$$

Lemma: For $w \in L^2(0,T;H^{-1/2}(\Gamma))$

$$c_S \, \|w\|_{L^2(0,T;H^{-1/2}(\Gamma))} \leq \sup_{0 \neq \phi \in V'_\Sigma} \frac{\langle Vw, \phi \rangle_\Sigma}{\|\phi\|_{V'_\Sigma}}$$

Boundary element mesh

$$\Sigma = \bigcup_{\ell=1}^{N} \overline{\sigma}_\ell, \quad \psi_\ell(x,t) = \begin{cases} 1 & \text{for } (x,t) \in \sigma_\ell, \\ 0 & \text{else} \end{cases}$$

Example

$$\alpha = 100, \quad u_0(x) = \sin \pi x \quad \text{for } x \in (0,1), \quad g(0,t) = g(1,t) = 0$$

L	N_{BEM}	Iter	$\|w - w_h\|_{L^2(0,T)}$	eoc
0	4	2	1.689 –1	
1	8	4	8.304 –2	1.024
2	16	8	4.113 –2	1.014
3	32	16	2.047 –2	1.007
4	64	27	1.021 –2	1.004
5	128	35	5.100 –3	1.001
6	256	42	2.550 –3	1.000
7	512	50	1.275 –3	1.000
8	1024	59	6.384 –4	0.998
9	2048	68	3.201 –4	0.996

Representation formula ($g \equiv 0$)

$$u(x,t) = \frac{1}{\alpha} \int_0^T \int_\Gamma U^*(x-y,t-s) w(y,s)\, ds_y ds + \int_\Omega U^*(x-y,t) u_0(y)\, dy$$

Approximate representation formula

$$\widetilde{u}(x,t) = \frac{1}{\alpha} \int_0^T \int_\Gamma U^*(x-y,t-s) w_h(y,s)\, ds_y ds + \int_\Omega U^*(x-y,t) u_0(y)\, dy$$

Define piecewise linear finite element interpolation

$$\widetilde{u}_h = I_h \widetilde{u}$$

Local error indicators

$$\eta_k = \|\widetilde{u} - I_h \widetilde{u}\|_{L^2(q_k)}$$

Adaptive finite element–boundary element solution [OS 1999]

Representation formula ($g \equiv 0$)

$$u(x,t) = \frac{1}{\alpha}\int_0^T \int_\Gamma U^*(x-y,t-s)w(y,s)\,ds_y ds + \int_\Omega U^*(x-y,t)u_0(y)\,dy$$

Normal derivative on Σ

$$\widetilde{w}(x,t) = \frac{1}{2}w_h(x,t) + \frac{1}{\alpha}\int_0^T \int_\Gamma w_h(y,s)\partial_{n_x} U^*(x-y,t-s)\,ds_y ds + \int_\Omega \partial_{n_x} U^*(x-y,t)u_0(y)dy$$

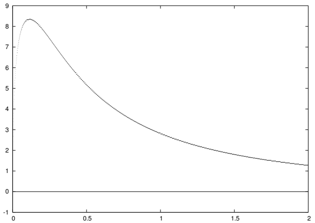
Error equation Laplace: [H. Schulz, OS 2000]

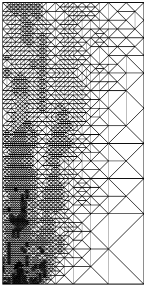
$$(\frac{1}{2}I - K')(w - w_h) = \widetilde{w} - w_h \quad \text{on } \Sigma$$

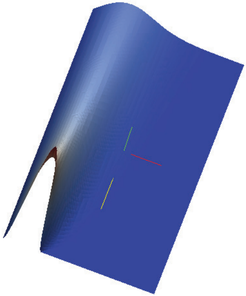
Error indicator

$$\widetilde{e}_h(x,t) = \widetilde{w}(x,t) - w_h(x,t)$$

L	N_{BEM}	$\|\widetilde{w} - w_h\|_{L^2(0,T)}$	eoc	N_{FEM}	M_{FEM}	$\|\widetilde{u} - \widetilde{u}_h\|_{L^2(Q)}$	eoc
uniform refinement							
0	4	1.016 –0		4	6	2.404 –1	
1	8	4.895 –1	1.054	16	15	2.130 –1	0.175
2	16	3.705 –1	0.402	64	45	1.558 –1	0.451
3	32	3.176 –1	0.222	256	153	2.932 –2	2.410
4	64	2.282 –1	0.477	1024	561	1.084 –2	1.436
5	128	1.468 –1	0.636	4096	2145	2.777 –3	1.965
6	256	8.836 –2	0.812	16384	8385	7.175 –4	1.952
adaptive refinement							
13	266	8.298 –3		5017	2573	7.786 –4	







Calderon projection

$$\begin{pmatrix} u \\ \partial_n u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ \partial_n u \end{pmatrix}$$

Identity

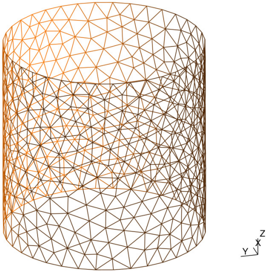
$$VD = \frac{1}{4}I - K^2$$

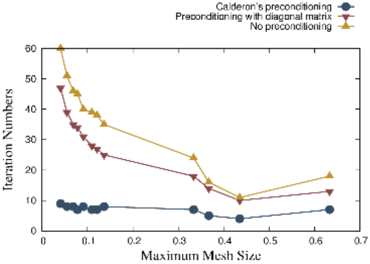
Operator preconditioning [OS, W. Wendland 1998]

Example: $\Omega = (0,1)$, $T = 1$, $g = 0$, $u_0(x) = \sin 2\pi x$

L	N	$C_V^{-1} = I$	$C_V^{-1} = M_h^{-1} V_h M_h^{-1}$
4	32	16	14
5	64	31	13
6	128	41	13
7	256	50	12
8	512	59	12
9	1024	70	11
10	2048	82	11
11	4096	96	10

Example: $\Omega = B_{1/2}(0),\ T = 1$ [K. Niino]





Transmission problem

$$\begin{aligned}
\alpha\partial_t u_i(x,t) - \operatorname{div}_x[A(x,t)\nabla_x u_i(x,t)] &= f(x,t) && \text{for } (x,t) \in \Omega \times (0,T),\\
\alpha\partial_t u_e(x,t) - \Delta u_e(x) &= 0 && \text{for } (x,t) \in \Omega^c \times (0,T),\\
u_i(x,0) &= u_0(x) && \text{for } x \in \Omega,\\
u_e(x,0) &= 0 && \text{for } x \in \Omega^c.
\end{aligned}$$

Transmission conditions for $(x,t) \in \Gamma \times (0,T)$:

$$u_i(x,t) = u_e(x,t), \quad n_x \cdot A(x,t)\nabla_x u_i(x,t) = \frac{\partial}{\partial n_x} u_e(x,t)$$

Radiation condition as $|x| \to \infty$, $t \in (0,T)$.

Non–symmetric BEM/FEM (Johnson–Nedelec) coupling

Variational problem

Find $u_i \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T,\widetilde{H}^{-1}(\Omega))$, $u_i(x,0) = u_0(x)$ for $x \in \Omega$, and $w \in [L^2_0(0,T;H^{1/2}(\Gamma))]'$ such that

$$\alpha \int_0^T \int_\Omega \partial_t u_i(x,t) v(x,t)\, dxdt + \int_0^T \int_\Omega [A(x,t)\nabla_x u_i(x,t)] \cdot \nabla_x v(x,t)\, dxdt \\ - \int_0^T \int_\Gamma w(x,t) v(x,t)\, ds_x dt = \int_0^T \int_\Omega f(x,t) v(x,t)\, dxdt$$

is satisfied for all $v \in L^2(0,T;H^1(\Omega))$, and for $(x,t) \in \Gamma \times (0,T)$:

$$\frac{1}{\alpha} \int_0^T \int_\Gamma U^*(x-y,t-s) w(y,s) ds_y ds \\ + \frac{1}{2} u_i(x,t) - \frac{1}{\alpha} \int_0^T \int_\Gamma \frac{\partial}{\partial n_y} U^*(x-y,t-s) u_i(y,s) ds_y ds_x = 0$$

Stability in the elliptic case: [Sayas 2009, 2013; OS 2011; G. Of, OS 2013]

Numerical example [S. Dohr 2015]

$$\alpha = 20, \quad T = 1, \quad u_0(x) = \begin{cases} \exp\left(\dfrac{1}{(2x-1)^2}\right) \sin \pi x & \text{for } x \in (0,1), \\ 0 & \text{else} \end{cases}$$

L	M_Ω	N_Ω	N_Σ	$\lVert u_i - u_{i,h} \rVert_{L^2(Q)}$	eoc
0	9	8	4	3.162 –2	
1	25	32	8	1.578 –2	1.003
2	81	128	16	4.223 –3	1.902
3	289	512	32	1.119 –3	1.915
4	1089	2048	64	2.896 –4	1.951
5	4225	8192	128	7.374 –5	1.974
6	16641	32768	256	1.872 –5	1.978
7	66049	131072	512	4.737 –6	1.983

Wave equation [M. Zank, OS 2016]

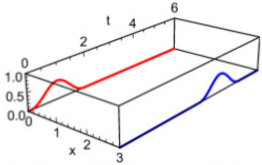


Fig. 1: Smooth Dirichlet datum $g = (g_0, g_L)^\top$ for $L = 3$ and $T = 6$.

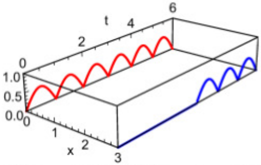


Fig. 2: Non-smooth Dirichlet datum $g = (g_0, g_L)^\top$ for $L = 3$ and $T = 6$.

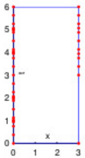


Fig. 3: Adaptive mesh for non-smooth solution.

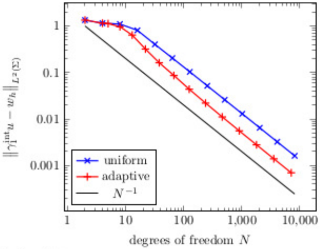


Fig. 4: $L^2(\Sigma)$ error for adaptive refinements for a smooth solution.

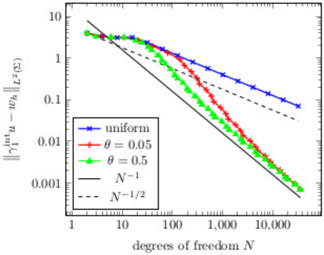


Fig. 5: $L^2(\Sigma)$ error for adaptive refinements for a non-smooth solution.

Conclusions

- space–time finite and boundary element methods
- arbitrary space–time meshes, no time stepping schemes
- adapativity in space and time simultaneously
- a posteriori error estimators
- parallel iterative solvers and preconditioners
- Fast BEM
- ...